Optimal decay estimates for the general solution to a class of semi-linear dissipative hyperbolic equations

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Abstract

We consider a class of semi-linear dissipative hyperbolic equations in which the operator associated to the linear part has a nontrivial kernel.

Under appropriate assumptions on the nonlinear term, we prove that all solutions decay to 0, as $t \to +\infty$, at least as fast as a suitable negative power of t. Moreover, we prove that this decay rate is optimal in the sense that there exists a nonempty open set of initial data for which the corresponding solutions decay exactly as that negative power of t.

Our results are stated and proved in an abstract Hilbert space setting, and then applied to partial differential equations.

Mathematics Subject Classification 2010 (MSC2010): 35B40, 35L71, 35L90.

Key words: semi-linear hyperbolic equation, dissipative hyperbolic equations, slow solutions, decay estimates, energy estimates.

1 Introduction

The present work has its origin in the search for decay estimates of solutions to some evolution equations of the general form

$$u''(t) + u'(t) + Au(t) + f(u(t)) = 0, \tag{1.1}$$

where H is a real Hilbert space, A is a nonnegative self-adjoint linear operator on H with dense domain, and f is a nonlinearity tangent to 0 at the origin.

When $f \equiv 0$, then for rather general classes of strongly positive operators A it is known that all solutions decay to 0 (as $t \to +\infty$) exponentially in the energy norm. Therefore, by perturbation theory it is reasonable to expect that also all solutions of (1.1) which decay to 0 have an exponential decay rate. The situation is different when A has a non-trivial kernel. In this case solutions tend to 0 if f fulfils suitable sign conditions, but we do not expect all solutions to have an exponential decay rate. Let us consider for example the hyperbolic equation

$$u_{tt} + u_t - \Delta u + |u|^p u = 0, \tag{1.2}$$

with homogeneous Neumann boundary conditions in a bounded domain Ω. In [12], by relying on the so-called Łojasiewicz gradient inequality [15, 16], it was established that, for any sufficiently small integer p, all solutions of this problem tend to 0 in the energy norm at least as fast as $t^{-1/p}$. Showing the optimality of this estimate means exhibiting a "slow solution", namely a solution decaying exactly as $t^{-1/p}$.

The existence of slow solutions for the Neumann problem was proved in [12] in the special case $p = 2$. The main idea is that each solution $v(t)$ to the ordinary differential equation

$$v'' + v' + |v|^p v = 0 \tag{1.3}$$

corresponds to the spatially homogeneous solution $u(t, x) := v(t)$ of (1.2), so that it is enough to exhibit a family of solutions of (1.3) decaying exactly as $t^{-1/2}$. It was later shown in [10] that actually any solution of (1.2) tends to 0 either exponentially or exactly as $t^{-1/p}$. This is the so-called "slow-fast alternative". Moreover at this occasion the set of initial data producing exponentially decaying solutions was shown to be closed with empty interior. In particular the set of "slow" solutions corresponds to an open set of initial data, but apart from the spatially homogeneous solutions no explicit condition on the initial data was found in [10].

The proofs of these results seem to exploit in an essential way the fact that the kernel of the linear part (in this case the set of constant functions) is an invariant space for (1.2). Without this assumption, both the alternative and the optimality of decay rates remained open problems.

Indeed let us consider, as a model case, the hyperbolic equation

$$u_{tt} + u_t - \Delta u - \lambda_1 u + |u|^p u = 0 \tag{1.4}$$

with homogeneous Dirichlet boundary conditions (here λ_1 denotes the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$). Now the kernel of the operator is the first eigenspace, which is not invariant by the nonlinear term, and even the existence of a slow solution decaying exactly as $t^{-1/p}$ was unknown until now.

In this paper we consider a general evolution equation of type (1.1), with f a gradient operator satisfying some regularity and structure conditions. Our aim is twofold. To begin with, in Theorem 2.2 we establish a general upper estimate of the energy, valid for all solutions. This estimate is proved in a quite general context through a modified Lyapunov functional, without any analyticity assumption on f. Then in Theorem 2.3 we prove the existence of slow solutions. This is the main result of this paper.

Our abstract theory applies to both (1.2) and (1.4). This shows in particular that the natural upper energy estimate for solutions of these problems is in general optimal, thereby settling an open problem raised in [12] and not solved, even for the special case (1.4), by the results of [10].

The problem of slow solutions has already been considered in the parabolic setting, and in particular in the case of equation

$$u_t - \Delta u + |u|^p u = 0 \tag{1.5}$$

with homogeneous Neumann boundary conditions in a bounded domain Ω, and in the case of equation

$$u_t - \Delta u - \lambda_1 u + |u|^p u = 0 \tag{1.6}$$

with homogeneous Dirichlet boundary conditions. In the case of (1.5), an easy application of the maximum principle shows that all solutions decay to 0 in $L^\infty(\Omega)$ at least as fast as $t^{-1/p}$ as $t \to +\infty$. The same property is true for (1.6) but more delicate to establish (see for example [13]). With Neumann boundary conditions, the optimality of this decay rate can be confirmed by looking at spatially homogenous solutions as in the hyperbolic setting. With Dirichlet boundary conditions, a comparison with suitable sub-solutions proves that all solutions with nonnegative initial data are actually slow solutions (see [13] for the details), which verifies the optimality of the upper estimate also in this second case. Moreover, in the case of Neumann boundary conditions, the slow-fast alternative is known (see [2]), fast solutions are known to be "exceptional", and some explicit classes of slow solutions with a sign changing initial datum were found in [3]. On the contrary, in the case of Dirichlet boundary conditions, even the slow-fast alternative is presently an open problem.

All results for these parabolic problems rely on the existence of special invariant sets, or on comparison arguments. Both tools do not extend easily to second order equations of the general form (1.1). For this reason, in this paper we follow a different path. The main idea is to look for slow solutions in the place where they are more likely to be, namely close to the kernel of A. Thus, under the assumption that $|f(u)| \sim |u|^{p+1}$, we look for solutions of (1.1) such that

$$\langle Au(t), u(t)\rangle \leq C|u(t)|^{2p+2} \qquad \forall t \geq 0 \tag{1.7}$$

for a suitable constant C. Roughly speaking, under this condition the term $Au(t)$ in (1.1) can be neglected, and the dynamical behavior is decided by the nonlinearity only. Thus we are in a situation analogous to the ordinary differential equation (1.3), for which the existence of slow solutions can be easily established. In order to prove (1.7), one is naturally led to consider the quotient

$$Q_p(t) := \frac{\langle Au(t), u(t)\rangle}{|u(t)|^{2p+2}},$$

which seems to be a p-extension of the Dirichlet quotient (the same quantity with $p = 0$), well known in many questions concerning parabolic problems (see for example the classical papers [1, 5] or the more recent [14]).

The Dirichlet quotient is nonincreasing in the case of linear homogeneous parabolic equations. This could naively lead to guess the monotonicity, or at least the boundedness, of $Q_p(t)$ also in the case of the second order problem (1.1). Of course this is not true as stated, but it is true for a hyperbolic version of $Q_p(t)$ with a kinetic term in the numerator. Thus we obtain the energy $G(t)$ defined by (3.22), which in turn we perturb by adding a mixing term, in such a way that the final energy $\widehat{G}(t)$ given by (3.23) satisfies a reasonable differential inequality. This strategy is inspired by similar modified Dirichlet quotients introduced in [6], and then largely exploited in [7, 8] in the context of Kirchhoff equations. In those papers the setting is different (quasi-linear instead of semi-linear), the goal is different (in [6] the main problem is the existence of global solutions), but the strategy is the same (comparing solutions of partial differential equations with solutions of ordinary differential equations), thus similar tools can be applied.

Our method produces not only some special slow solution, but an open set in the basic energy space. This is the first step towards proving that slow solutions are in some sense generic, in accordance with the general idea that the slowest decay rate is dominant, and faster solutions are somewhat atypical. We plan to consider this issue in a future research.

This paper is organized as follows. In section 2 we clarify the functional setting, we recall the notion of weak solutions, and we state our main abstract results. In section 3 we prove them. In section 4 we present some applications of our theory to dissipative hyperbolic equations.

2 Functional setting and main abstract results

We consider the semilinear abstract second order equation

$$u''(t) + u'(t) + Au(t) + \nabla F(u(t)) = 0 \qquad \forall t \geq 0, \tag{2.1}$$

with initial data

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2.2}$$

We always assume that H is a Hilbert space, and A is a self-adjoint linear operator on H with dense domain $D(A)$. We assume that A is nonnegative, namely $\langle Au, u\rangle \geq 0$ for every $u \in D(A)$, so that for every $\alpha \geq 0$ the power $A^\alpha u$ is defined provided that u lies in a suitable domain $D(A^\alpha)$, which is itself a Hilbert space with norm

$$|u|_{D(A^\alpha)} := \left(|u|^2 + |A^\alpha u|^2\right)^{1/2}.$$

We assume that $F : D(A^{1/2}) \to \mathbb{R}$. When we write $\nabla F(u)$, we mean that there exists a function $\nabla F : D(A^{1/2}) \to H$ such that

$$\lim_{|v|_{D(A^{1/2})} \to 0} \frac{F(u+v) - F(u) - \langle \nabla F(u), v\rangle}{|v|} = 0 \qquad \forall u \in D(A^{1/2}). \tag{2.3}$$

The existence of $\nabla F(u)$ in the sense of (2.3) is enough to guarantee the continuity of F with respect to the norm of $D(A^{1/2})$. Moreover, for every $u \in C^1([0,+\infty); H) \cap C^0([0,+\infty); D(A^{1/2}))$ we have that the function $t \to F(u(t))$ is of class C^1, and its time-derivative can be computed with the usual chain rule

$$\frac{d}{dt}\left[F(u(t))\right] = \langle \nabla F(u(t)), u'(t)\rangle \qquad \forall t \geq 0.$$

We always assume that $\nabla F : D(A^{1/2}) \to H$ is locally Lipschitz continuous, namely

$$|\nabla F(u) - \nabla F(v)| \leq L\left(|u|_{D(A^{1/2})}, |v|_{D(A^{1/2})}\right) \cdot |u - v|_{D(A^{1/2})} \tag{2.4}$$

for every u and v in $D(A^{1/2})$, for a suitable function $L : \mathbb{R}^2 \to \mathbb{R}$ which is bounded on bounded sets. Under these hypotheses, one obtains the following result concerning global existence, regularity and derivatives of energies.

Proposition 2.1 *Let H be a Hilbert space, let A be a self-adjoint nonnegative operator on H with dense domain $D(A)$, and let $F : D(A^{1/2}) \to \mathbb{R}$.*

Let us assume that

(i) $F(u) \geq 0$ for every $u \in D(A^{1/2})$,

(ii) F has a gradient $\nabla F : D(A^{1/2}) \to H$ in the sense of (2.3),

(iii) ∇F is locally Lipschitz continuous in the sense of (2.4).

Then, for every $(u_0, u_1) \in D(A^{1/2}) \times H$, problem (2.1)–(2.2) admits a unique global weak solution

$$u \in C^0\left([0,+\infty); D(A^{1/2})\right) \cap C^1\left([0,+\infty); H\right). \tag{2.5}$$

In addition the functions

$$E_0(t) := \frac{1}{2}\left(|u'(t)|^2 + |A^{1/2}u(t)|^2\right), \qquad F_0(t) := E_0(t) + F(u(t)) \tag{2.6}$$

are of class C^1, and their time-derivative is given by

$$E_0'(t) = -|u'(t)|^2 - \langle \nabla F(u(t), u'(t)\rangle, \qquad F_0'(t) = -|u'(t)|^2. \tag{2.7}$$

The first main result of this paper is an upper energy estimate, valid for all weak solutions of (2.1).

Theorem 2.2 (Upper decay estimate for weak solutions) *Let us assume that*

(Hp1) H is a Hilbert space, and A is a self-adjoint nonnegative operator on H with dense domain $D(A)$,

(Hp2) $F : D(A^{1/2}) \to [0,+\infty)$ is a nonnegative function with $F(0) = 0$,

(Hp3) F has a gradient $\nabla F : D(A^{1/2}) \to H$ in the sense of (2.3),

(Hp4) ∇F is locally Lipschitz continuous in the sense of (2.4),

(Hp5) there exists a constant $K > 0$ such that

$$\langle \nabla F(u), u\rangle \geq K \cdot F(u) \qquad \forall u \in D(A^{1/2}), \tag{2.8}$$

(Hp6) there exist $p > 0$, and a function $R_1 : \mathbb{R} \to \mathbb{R}$ which is bounded on bounded sets, such that

$$|u|^{p+2} \leq R_1(|u|_{D(A^{1/2})}) \cdot \left(|A^{1/2}u|^2 + F(u)\right) \qquad \forall u \in D(A^{1/2}). \tag{2.9}$$

Let $(u_0, u_1) \in D(A^{1/2}) \times H$, and let $u(t)$ be the unique global weak solution of problem (2.1)–(2.2) provided by Proposition 2.1.

Then there exist constants M_1 and M_2 such that

$$|u'(t)|^2 + |A^{1/2}u(t)|^2 + F(u(t)) \leq \frac{M_1}{(1+t)^{1+2/p}} \qquad \forall t \geq 0, \tag{2.10}$$

$$|u(t)| \leq \frac{M_2}{(1+t)^{1/p}} \qquad \forall t \geq 0. \tag{2.11}$$

Our second main result is the existence of an open set of slow solutions, namely solutions for which (2.11) is optimal.

Theorem 2.3 (Existence of slow solutions) *Let us assume that hypotheses (Hp1) through (Hp4) of Theorem 2.2 are satisfied. In addition, let us assume that*

$$\ker A \neq \{0\}, \tag{2.12}$$

$$\exists \nu > 0 \text{ such that } |A^{1/2}u|^2 \geq \nu |u|^2 \quad \forall u \in D(A^{1/2}) \cap \ker(A)^{\perp}, \tag{2.13}$$

and that there exist real numbers $\rho > 0$, $R > 0$, $\alpha > 0$ such that

$$|\nabla F(u)| \leq R\left(|u|^{p+1} + |A^{1/2}u|^{1+\alpha}\right) \tag{2.14}$$

for every $u \in D(A^{1/2})$ with $|u|_{D(A^{1/2})} \leq \rho$.

Then there exist a nonempty open set $\mathcal{S} \subseteq D(A^{1/2}) \times H$ and a constant M_3 such that, for every $(u_0, u_1) \in \mathcal{S}$, the unique global solution of problem (2.1)–(2.2) provided by Proposition 2.1 satisfies

$$|u(t)| \geq \frac{M_3}{(1+t)^{1/p}} \qquad \forall t \geq 0. \tag{2.15}$$

Remark 2.4 Condition (2.13) is known to be equivalent to the property that A has closed range $R(A) = (\ker A)^{\perp}$.

Let $P : H \to \ker A$ denote the orthogonal projection on $\ker A$, and let $Q = I - P$ denote the orthogonal projection on $R(A)$. From (2.13) and (2.10) it follow that

$$|Qu(t)|^2 \leq \frac{1}{\nu}|A^{1/2}u(t)|^2 \leq \frac{M_1}{\nu}\frac{1}{(1+t)^{1+2/p}}.$$

Since $|u|^2 = |Pu|^2 + |Qu|^2$ for every $u \in H$, comparing with (2.15) we obtain that there exists a constant M_4 such that

$$|Pu(t)| \geq \frac{M_4}{(1+t)^{1/p}} \qquad \forall t \geq 0.$$

In other words, the range component decays faster, and the slow decay of $u(t)$ is due to its component with respect to $\ker A$. This extends to the general abstract setting what previously observed in the special case studied in [10].

3 Proofs

3.1 Proof of Proposition 2.1

Local existence We consider the Hilbert space $\mathcal{H} := D(A^{1/2}) \times H$, endowed with the norm defined by

$$|U|_{\mathcal{H}}^2 = |(u,v)|_{\mathcal{H}}^2 = |u|_{D(A^{1/2})}^2 + |v|^2,$$

the subspace $D(\mathcal{A}) := D(A) \times D(A^{1/2})$, the linear operator

$$\mathcal{A}(u,v) := (-v, Au + u) \qquad \forall (u,v) \in D(\mathcal{A}),$$

and the operator

$$\mathcal{F}(u,v) := (0, u - \nabla F(u) - v) \qquad \forall (u,v) \in \mathcal{H}.$$

It is easy to check that $\mathcal{A}$ is a skew-adjoint linear operator, hence in particular a maximal monotone linear operator on $\mathcal{H}$ with dense domain $D(\mathcal{A})$, and $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is a locally Lipschitz continuous operator. Introducing $U(t) := (u(t), u'(t))$, one can rewrite problem (2.1)–(2.2) in the form

$$U'(t) + \mathcal{A}U(t) = \mathcal{F}(U(t)) \qquad \forall t \geq 0,$$

with initial datum $U(0) = U_0 := (u_0, u_1)$. Thus we have reduced our problem to the framework of Lipschitz perturbations of maximal monotone operators. At this point, local existence follows from classical results, for which we refer to Theorem 4.3.4 and Proposition 4.3.9 of [4]. More precisely, we obtain the following.

- *(Local existence of weak solutions)* For every $(u_0, u_1) \in D(A^{1/2}) \times H$, there exists $T > 0$ such that problem (2.1)–(2.2) has a unique weak solution

$$u \in C^0\left([0,T); D(A^{1/2})\right) \cap C^1\left([0,T); H\right).$$

- *(Continuation)* The local solution can be continued to a solution defined in a maximal interval $[0, T_*)$, with either $T_* = +\infty$, or

$$\limsup_{t \to T_*^-} \left(|u'(t)|_H^2 + |u(t)|_{D(A^{1/2})}^2 \right) = +\infty.$$

Differentiation of energies We show that for all weak solutions the functions $E_0(t)$ and $F_0(t)$ defined by (2.6) are of class C^1, and their time-derivative is given by (2.7) for every $t \in [0, T)$. Indeed for the first result we can consider the isometry group generated on $\mathcal{H}$ by $\mathcal{A}$. Then Lemma 11 of [9] (see also [17] for an earlier more general result in the same direction) gives

$$E_0'(t) + \langle u(t), u'(t) \rangle = \langle \mathcal{F}(U(t)), U(t) \rangle_{\mathcal{H}} = \langle u(t) - \nabla F(u) - u'(t), u'(t) \rangle$$

yielding the proper result for E_0. The result for F_0 follows also since $\langle \nabla F(u(t)), u'(t) \rangle$ is the derivative of the C^1 function $F(u(t))$ as a consequence of the chain rule, as already observed.

Global existence Thanks to the "continuation" result, all we need to show is that $E_0(t)$ is bounded uniformly in time. This follows at once from the nonincreasing character of F_0 and our assumption that $F(u) \geq 0$.

3.2 A basic a priori estimate

The next simple a priori estimate will be useful in the proof of both main theorems.

Proposition 3.1 *Let H be a Hilbert space, let A be a self-adjoint nonnegative operator on H with dense domain $D(A)$, and let $F : D(A^{1/2}) \to \mathbb{R}$.*

Let us assume that

(i) $F(u) \geq 0$ *for every* $u \in D(A^{1/2})$,

(ii) F *has a gradient* $\nabla F : D(A^{1/2}) \to H$ *in the sense of (2.3),*

(iii) $\langle \nabla F(u), u \rangle \geq 0$ *for every* $u \in D(A^{1/2})$.

Let $(u_0, u_1) \in D(A^{1/2}) \times H$, *and let* $u(t)$ *be the local weak solution of problem (2.1)–(2.2) in some time-interval* $[0, T)$. *Then we have*

$$|u'(t)|^2 + |u(t)|^2 + |A^{1/2}u(t)|^2 + F(u(t)) \leq 16 \left(|u_1|^2 + |u_0|^2 + |A^{1/2}u_0|^2 + F(u_0)\right) \quad (3.1)$$

for every $t \in [0, T)$.

Proof Let us consider the two different energies

$$\widetilde{E}(t) := |u'(t)|^2 + \frac{1}{2}|u(t)|^2 + |A^{1/2}u(t)|^2 + 2F(u(t)) + \langle u'(t), u(t) \rangle,$$

$$\widehat{E}(t) := |u'(t)|^2 + |u(t)|^2 + |A^{1/2}u(t)|^2 + F(u(t)).$$

Due to assumption (i) and inequality

$$|\langle u'(t), u(t) \rangle| \leq \frac{3}{8}|u(t)|^2 + \frac{2}{3}|u'(t)|^2,$$

it is easy to see that

$$\frac{1}{8}\widehat{E}(t) \leq \widetilde{E}(t) \leq 2\widehat{E}(t) \qquad \forall t \in [0, T). \quad (3.2)$$

The function $\widetilde{E}(t)$ is of class C^1, even in the case of weak solutions, and its time-derivative is

$$\widetilde{E}'(t) = -|u'(t)|^2 - |A^{1/2}u(t)|^2 - \langle \nabla F(u(t)), u(t) \rangle.$$

From assumption (iii) we see that $\widetilde{E}'(t) \leq 0$, hence $\widetilde{E}(t) \leq \widetilde{E}(0)$ for every $t \in [0, T)$. Keeping (3.2) into account, we have proved that

$$\widehat{E}(t) \leq 8\widetilde{E}(t) \leq 8\widetilde{E}(0) \leq 16\widehat{E}(0) \qquad \forall t \in [0, T),$$

which is exactly (3.1). □

3.3 Proof of Theorem 2.2

Let us describe the strategy of the proof before entering into details. We consider the energies

$$E(t) := |u'(t)|^2 + |A^{1/2}u(t)|^2 + 2F(u(t)) = 2F_0(t), \tag{3.3}$$

$$\widehat{E}_\varepsilon(t) := E(t) + \varepsilon\,[E(t)]^\beta\,\langle u'(t), u(t)\rangle, \tag{3.4}$$

where $\varepsilon > 0$ is a parameter and

$$\beta := \frac{p}{p+2}. \tag{3.5}$$

Now we claim three facts (from now on, all positive constants ε_0, ε_1, c_0, …, c_{10} depend on p, $|u_0|$, $E(0)$, K, and on the function R_1).

- *First claim.* There exist c_0 and c_1 such that

$$E(t) \le c_0 \qquad \forall t \ge 0, \tag{3.6}$$

$$|u(t)|^{p+2} \le c_1 E(t) \qquad \forall t \ge 0. \tag{3.7}$$

- *Second claim.* There exists $\varepsilon_0 > 0$ such that

$$\frac{1}{2}E(t) \le \widehat{E}_\varepsilon(t) \le 2E(t) \qquad \forall t \ge 0,\ \forall \varepsilon \in (0, \varepsilon_0]. \tag{3.8}$$

- *Third claim.* There exist $\varepsilon_1 \in (0, \varepsilon_0]$, and a constant $c_2 > 0$, such that

$$\widehat{E}'_\varepsilon(t) \le -c_2\varepsilon\left[\widehat{E}_\varepsilon(t)\right]^{1+\beta} \qquad \forall t \ge 0,\ \forall \varepsilon \in (0, \varepsilon_1]. \tag{3.9}$$

If we prove these three claims, then we easily obtain (2.10) and (2.11). Indeed let us integrate the differential inequality (3.9) with $\varepsilon = \varepsilon_1$. We obtain the inequality

$$\widehat{E}_{\varepsilon_1}(t) \le \frac{c_3}{(1+t)^{1/\beta}} \qquad \forall t \ge 0.$$

Thanks to (3.8) and (3.5), this proves (2.10). At this point, (2.11) follows from (2.10) and (3.7). So we are left to proving our three claims.

Proof of first claim We can apply Proposition 3.1. Thus obtain estimate (3.6) and the boundedness of $|u(t)|_{D(A^{1/2})}$. Then (3.7) follows from (3.6) and assumption (2.9).

Proof of second claim From (3.7) and (3.3) we have

$$|\langle u'(t), u(t)\rangle| \le |u'(t)| \cdot |u(t)| \le [E(t)]^{1/2} \cdot c_4\,[E(t)]^{1/(p+2)}, \tag{3.10}$$

hence

$$[E(t)]^\beta\,|\langle u'(t), u(t)\rangle| \le c_4\,[E(t)]^{p/(2p+4)} \cdot E(t).$$

Since $p > 0$, with the help of (3.6) we deduce

$$[E(t)]^\beta\,|\langle u'(t), u(t)\rangle| \le c_5 E(t) \qquad \forall t \ge 0.$$

This implies that (3.8) holds true provided that $c_5\varepsilon_0 \le 1/2$.

Proof of third claim The time-derivative of (3.4) is

$$\begin{aligned}
\widehat{E}'_\varepsilon(t) &= -2|u'(t)|^2 - 2\varepsilon\beta\,[E(t)]^{-2/(p+2)}\,\langle u'(t),u(t)\rangle|u'(t)|^2 + \varepsilon\,[E(t)]^{\beta}\,|u'(t)|^2 \\
&\quad -\varepsilon\,[E(t)]^{\beta}\,\langle u'(t),u(t)\rangle - \varepsilon\,[E(t)]^{\beta}\left(|A^{1/2}u(t)|^2 + \langle \nabla F(u(t)),u(t)\rangle\right) \\
&=: F_1+F_2+F_3+F_4+F_5. \qquad (3.11)
\end{aligned}$$

Let us estimate separately the sum F_2+F_3 and the two last terms. First (3.10) implies

$$[E(t)]^{-2/(p+2)}\,|\langle u'(t),u(t)\rangle| \le c_4\,[E(t)]^{p/(2p+4)}\,.$$

Then, since $p>0$, by using (3.6) we derive

$$F_2+F_3 \le \varepsilon|u'(t)|^2\left(c_6\,[E(t)]^{p/(2p+4)} + [E(t)]^{\beta}\right) \le c_7\varepsilon|u'(t)|^2. \qquad (3.12)$$

Moreover from (3.7) and (3.5) we infer

$$\varepsilon\,[E(t)]^{\beta}\,|\langle u'(t),u(t)\rangle| \le \frac{1}{2}|u'(t)|^2 + \frac{1}{2}\varepsilon^2\,[E(t)]^{2\beta}\,|u(t)|^2 \le \frac{1}{2}|u'(t)|^2 + c_8\varepsilon^2\,[E(t)]^{2\beta+2/(p+2)}\,.$$

Since $2\beta+2/(p+2)=\beta+1$, this means that

$$F_4 \le \frac{1}{2}|u'(t)|^2 + c_8\varepsilon^2\,[E(t)]^{\beta+1}\,. \qquad (3.13)$$

Finally, from assumption (2.8) and the inequality (3.6), we deduce

$$\begin{aligned}
[E(t)]^{\beta}\left(|A^{1/2}u(t)|^2 + \langle \nabla F(u(t)),u(t)\rangle\right) &\ge c_9\,[E(t)]^{\beta}\left(|A^{1/2}u(t)|^2 + F(u(t))\right) \\
&\ge \frac{c_9}{2}\,[E(t)]^{\beta}\left(E(t) - |u'(t)|^2\right) \\
&\ge c_{10}\,[E(t)]^{\beta+1} - c_{11}|u'(t)|^2,
\end{aligned}$$

hence

$$F_5 \le -c_{10}\varepsilon\,[E(t)]^{\beta+1} + c_{11}\varepsilon|u'(t)|^2. \qquad (3.14)$$

Plugging (3.12) through (3.14) into (3.11), we now find

$$\widehat{E}'_\varepsilon(t) \le -|u'(t)|^2\left(\frac{3}{2} - c_7\varepsilon - c_{11}\varepsilon\right) + \varepsilon\,(c_8\varepsilon - c_{10})\,[E(t)]^{\beta+1}\,.$$

If we choose $\varepsilon_1 \in (0,\varepsilon_0]$ small enough so that

$$c_7\varepsilon_1 + c_{11}\varepsilon_1 \le \frac{3}{2} \qquad \text{and} \qquad c_8\varepsilon_1 - c_{10} \le -\frac{c_{10}}{2},$$

then (3.9) holds true with $c_2 = c_{10}/2 > 0$. This completes the proof of Theorem 2.2 $\square$

3.4 Proof of Theorem 2.3

Let us describe the strategy of the proof before entering into details. Let ν, ρ, R, α be the constants appearing in (2.13) and (2.14). First of all, let us choose $\delta > 0$ such that

$$\delta \leq \frac{\nu}{2\nu + 1}. \tag{3.15}$$

Note that this condition implies in particular that

$$\delta \leq 1 \qquad \text{and} \qquad \delta \leq \frac{\sqrt{\nu}}{2}. \tag{3.16}$$

Let Q denote the orthogonal projection from H to $(\ker A)^{\perp}$. Assuming $(u_0, u_1) \in D(A^{1/2}) \times H$ and $u_0 \neq 0$, we set

$$\sigma_0 := 4\left(|u_1|^2 + |u_0|^2 + |A^{1/2}u_0|^2 + F(u_0)\right)^{1/2},$$

$$\sigma_1 := \frac{1}{|u_0|^{2p+2}}\left(\frac{1}{2}|u_1|^2 + \frac{1}{2}|A^{1/2}u_0|^2 + \delta|\langle u_1, Qu_0\rangle|\right) + \frac{128R^2}{\delta^2}.$$

Let $\mathcal{S} \subseteq D(A) \times D(A^{1/2})$ be the set of initial data such that

$$\sigma_0 < \rho, \qquad 2\sigma_0^{\alpha}R < \frac{\delta}{4}, \qquad 4(p+1)\sigma_0^{p}\sqrt{\sigma_1} < \frac{\delta}{32}. \tag{3.17}$$

It is clear that these smallness assumptions define an open set. This open set is nonempty because it contains at least all pairs (u_0, u_1) with $u_1 = 0$ and $u_0 \in \ker A$ with $u_0 \neq 0$ and $|u_0|$ small enough. This is the point where assumption (2.12) and the fact that $F(0) = 0$ are essential.

Now we claim that, for every pair of initial data $(u_0, u_1) \in \mathcal{S}$, the global weak solution of (2.1)–(2.2) satisfies

$$u(t) \neq 0 \qquad \forall t \geq 0, \tag{3.18}$$

and

$$\frac{1}{2}\frac{|u'(t)|^2 + |A^{1/2}u(t)|^2}{|u(t)|^{2p+2}} \leq 2\sigma_1 \qquad \forall t \geq 0. \tag{3.19}$$

This is enough to prove (2.15). Indeed, setting $y(t) := |u(t)|^2$, we observe that

$$|y'(t)| = 2|\langle u'(t), u(t)\rangle| \leq 2\frac{|u'(t)|}{|u(t)|^{1+p}} \cdot |u(t)|^{2+p} \leq 4\sqrt{\sigma_1} \cdot |y(t)|^{1+p/2}, \tag{3.20}$$

and in particular

$$y'(t) \geq -4\sqrt{\sigma_1} \cdot |y(t)|^{1+p/2} \qquad \forall t \geq 0. \tag{3.21}$$

Since $y(0) > 0$, this inequality concludes the proof. So we are left to prove (3.18) and (3.19). To this end, we set

$$G(t) := \frac{1}{2}\frac{|u'(t)|^2 + |A^{1/2}u(t)|^2}{|u(t)|^{2p+2}}, \tag{3.22}$$

and

$$T := \sup \left\{ t \geq 0 : \forall \tau \in [0,t],\ \ u(\tau) \neq 0 \text{ and } G(\tau) \leq 2\sigma_1 \right\}.$$

Since $u(0) \neq 0$, and $G(0) < \sigma_1$ (because of our definition of σ_1), we have that $T > 0$. We claim that $T = +\infty$, which is equivalent to (3.18) and (3.19). Let us assume by contradiction that this is not the case. Due to the maximality of T, this means that either $u(T) = 0$ or $G(T) = 2\sigma_1$. Now we show that both choices lead to an impossibility.

Let us set as usual $y(t) := |u(t)|^2$. For every $t \in [0,T)$ we have that $u(t) \neq 0$ and $G(t) \leq 2\sigma_1$. Therefore, arguing as in (3.20), we obtain that the differential inequality in (3.21) holds true for every $t \in [0,T)$. Since $y(0) > 0$, and $1 + p/2 \geq 1$, this differential inequality implies that $y(T) \neq 0$, hence $u(T) \neq 0$.

So it remains to show that $G(T) < 2\sigma_1$. To this end, we introduce the perturbed energy

$$\widehat{G}(t) := \frac{1}{2}\frac{|u'(t)|^2 + |A^{1/2}u(t)|^2}{|u(t)|^{2p+2}} + \delta\frac{\langle u'(t), Qu(t)\rangle}{|u(t)|^{2p+2}}. \tag{3.23}$$

Due to the second condition in (3.16), the energy $\widehat{G}(t)$ is a small perturbation of $G(t)$ in the sense that

$$\frac{1}{2}G(t) \leq \widehat{G}(t) \leq 2G(t) \qquad \forall t \in [0,T). \tag{3.24}$$

The correcting term $\langle u'(t), Qu(t)\rangle$ appears frequently when looking for boundedness or decay properties for equations whose generator has a non-trivial kernel (see [18] or [11]).

The time-derivative of $\widehat{G}$ is

$$\begin{aligned} \widehat{G}'(t) &= -\frac{|u'(t)|^2}{|u(t)|^{2p+2}} - \delta\frac{|A^{1/2}u(t)|^2}{|u(t)|^{2p+2}} - \frac{\langle \nabla F(u(t)), u'(t) + \delta Qu(t)\rangle}{|u(t)|^{2p+2}} \\ &\quad +\delta\frac{|Qu'(t)|^2 - \langle u'(t), Qu(t)\rangle}{|u(t)|^{2p+2}} - 2(p+1)\frac{\langle u'(t), u(t)\rangle}{|u(t)|^2}\cdot\widehat{G}(t) \\ &=: I_1 + \ldots + I_5. \end{aligned} \tag{3.25}$$

Let us estimate I_3, I_4, and I_5. First of all, from Proposition 3.1 we obtain that

$$|u(t)|^2 + |A^{1/2}u(t)|^2 \leq \sigma_0^2 \qquad \forall t \geq 0. \tag{3.26}$$

Therefore, from the first smallness condition in (3.17) and assumption (2.14), it follows that

$$|\nabla F(u(t))| \leq R\left(|u(t)|^{p+1} + |A^{1/2}u(t)|^{1+\alpha}\right) \leq R\left(|u(t)|^{p+1} + |A^{1/2}u(t)|\cdot\sigma_0^\alpha\right),$$

hence

$$\frac{|\nabla F(u(t))|}{|u(t)|^{p+1}} \leq R\left(1 + \frac{|A^{1/2}u(t)|}{|u(t)|^{p+1}}\cdot\sigma_0^\alpha\right) \leq R\left(1 + \sqrt{2G(t)}\cdot\sigma_0^\alpha\right). \tag{3.27}$$

On the other hand, from assumption (2.13) and the fact that $\delta \leq \sqrt{\nu}$, it follows that

$$\delta|Qu(t)| \leq \frac{\delta}{\sqrt{\nu}}|A^{1/2}u(t)| \leq |A^{1/2}u(t)|,$$

hence

$$\frac{|u'(t)| + \delta|Qu(t)|}{|u(t)|^{p+1}} \leq \frac{|u'(t)| + |A^{1/2}u(t)|}{|u(t)|^{p+1}} \leq \sqrt{2G(t)}. \tag{3.28}$$

From (3.27) and (3.28) it follows that

$$I_3 \leq \frac{|\nabla F(u(t))|}{|u(t)|^{p+1}} \cdot \frac{|u'(t)| + \delta|Qu(t)|}{|u(t)|^{p+1}} \leq R\sqrt{2G(t)} + 2R\sigma_0^\alpha G(t) \leq \frac{4R^2}{\delta} + \frac{\delta}{8}G(t) + 2R\sigma_0^\alpha G(t).$$

From the second smallness assumption in (3.17) we finally conclude that

$$I_3 \leq \frac{4R^2}{\delta} + \frac{3\delta}{8}G(t) \qquad \forall t \in [0,T). \tag{3.29}$$

As for I_4, we exploit that $|Qu'(t)| \leq |u'(t)|$ and $|Qu(t)| \leq \nu^{-1/2}|A^{1/2}u(t)|$, hence

$$\begin{aligned} |Qu'(t)|^2 + |u'(t)| \cdot |Qu(t)| \quad &\leq \quad |u'(t)|^2 + \frac{1}{\sqrt{\nu}}|u'(t)| \cdot |A^{1/2}u(t)| \\ &\leq \quad \left(1 + \frac{1}{2\nu}\right)|u'(t)|^2 + \frac{1}{2}|A^{1/2}u(t)|^2. \end{aligned}$$

Thus from (3.15) we deduce that

$$I_4 \leq \delta\frac{|Qu'(t)|^2 + |u'(t)| \cdot |Qu(t)|}{|u(t)|^{2p+2}} \leq \frac{1}{2}\frac{|u'(t)|^2}{|u(t)|^{2p+2}} + \frac{\delta}{2}\frac{|A^{1/2}u(t)|^2}{|u(t)|^{2p+2}} \qquad \forall t \in [0,T). \tag{3.30}$$

In order to estimate I_5, we exploit once again (3.26) and we obtain

$$I_5 \leq 2(p+1)\frac{|u'(t)|}{|u(t)|^{p+1}} \cdot |u(t)|^p \cdot \widehat{G}(t) \leq 2(p+1)\sqrt{2G(t)} \cdot \sigma_0^p \cdot \widehat{G}(t).$$

Since $G(t) \leq 2\sigma_1$ for every $t \in [0,T)$, the third smallness condition in (3.17) gives

$$I_5 \leq 4(p+1)\sqrt{\sigma_1} \cdot \sigma_0^p \cdot \widehat{G}(t) \leq \frac{\delta}{32}\widehat{G}(t) \qquad \forall t \in [0,T). \tag{3.31}$$

Plugging (3.29) through (3.31) into (3.25) we obtain

$$\widehat{G}'(t) \leq -\frac{1}{2}\frac{|u'(t)|^2}{|u(t)|^{2p+2}} - \frac{\delta}{2}\frac{|A^{1/2}u(t)|^2}{|u(t)|^{2p+2}} + \frac{4R^2}{\delta} + \frac{3\delta}{8}G(t) + \frac{\delta}{32}\widehat{G}(t).$$

Due to the first inequality in (3.16), this implies

$$\widehat{G}'(t) \leq -\frac{\delta}{2}G(t) + \frac{4R^2}{\delta} + \frac{3\delta}{8}G(t) + \frac{\delta}{32}\widehat{G}(t) = -\frac{\delta}{8}G(t) + \frac{4R^2}{\delta} + \frac{\delta}{32}\widehat{G}(t),$$

hence by (3.24)

$$\widehat{G}'(t) \leq -\frac{\delta}{32}\widehat{G}(t) + \frac{4R^2}{\delta} \qquad \forall t \in [0,T).$$

Integrating this differential inequality we easily deduce that

$$\widehat{G}(t) \leq \left(\widehat{G}(0) - \frac{128R^2}{\delta^2}\right)\exp\left(-\frac{\delta}{32}t\right) + \frac{128R^2}{\delta^2} \qquad \forall t \in [0,T). \tag{3.32}$$

Since we already know that $u(T) \neq 0$, we have that $G(t)$ and $\widehat{G}(t)$ are defined and continuous at least up to $t = T$. Letting $t \to T^-$ in (3.32), and exploiting (3.24) and our definition of σ_1, we deduce that

$$G(T) \leq 2\widehat{G}(T) < 2\left(\widehat{G}(0) + \frac{128R^2}{\delta^2}\right) \leq 2\sigma_1.$$

This excludes that $G(T) = 2\sigma_1$, thus completing the proof. $\square$

4 Applications to partial differential equations

4.1 Some equations with a local nonlinearity of power type

The following statement represents a bridge between the abstract theory and partial differential equations. Here H is a space of real valued functions, and we explicitly write $|u|_H$ for the norm of the function $u \in H$ (not to be confused with the absolute value $|u|$ of the same function). Now the abstract assumptions on ∇F are replaced by suitable inequalities between norms, which are going to become Sobolev type inequalities in the concrete settings.

Theorem 4.1 (Semi-abstract result for local equations) *Let $\mathbb{X}$ be a set and μ be a measure in $\mathbb{X}$ with $\mu(\mathbb{X}) < +\infty$. Let $H := L^2(\mathbb{X},\mu)$, and let A be a linear operator on H with dense domain $D(A)$ satisfying assumptions (2.12) and (2.13) of Theorem 2.3. Let $p > 0$, and let us consider the second order equation*

$$u''(t) + u'(t) + Au(t) + |u(t)|^p u(t) = 0. \tag{4.1}$$

Let us assume that

(i) $D(A^{1/2}) \subseteq L^{2(p+1)}(\mathbb{X},\mu)$, and there exists a constant K_1 such that

$$\|u\|_{L^{2(p+1)}(\mathbb{X},\mu)} \leq K_1 |u|_{D(A^{1/2})} \qquad \forall u \in D(A^{1/2}), \tag{4.2}$$

(ii) there exists a constant K_2 such that

$$\left\||u|^p v^2\right\|_{L^1(\mathbb{X},\mu)} \leq K_2 |u|^p_{D(A^{1/2})} \cdot |v|_{D(A^{1/2})} \cdot |v|_H \qquad \forall (u,v) \in \left[D(A^{1/2})\right]^2, \tag{4.3}$$

$$\left\||u|^{2p} v^2\right\|_{L^1(\mathbb{X},\mu)} \leq K_2 |u|^{2p}_{D(A^{1/2})} \cdot |v|^2_{D(A^{1/2})} \qquad \forall (u,v) \in \left[D(A^{1/2})\right]^2. \tag{4.4}$$

Then we have the following conclusions.

(1) (Decay for all weak solutions) *For every $(u_0, u_1) \in D(A^{1/2}) \times H$, problem (4.1), (2.2) has a unique global weak solution with the regularity prescribed by (2.5). Moreover there exists a constant M_1 such that*

$$\|u(t)\|_{L^2(\mathbb{X},\mu)} \leq \frac{M_1}{(1+t)^{1/p}} \qquad \forall t \geq 0.$$

(2) (Existence of slow solutions) *There exist a nonempty open set $\mathcal{S} \subseteq D(A^{1/2}) \times H$, and positive constants M_2 and M_3, with the following property. For every pair of initial conditions $(u_0, u_1) \in \mathcal{S}$, the unique global solution of problem (4.1), (2.2) satisfies*

$$\frac{M_2}{(1+t)^{1/p}} \leq \|u(t)\|_{L^2(\mathbb{X},\mu)} \leq \frac{M_3}{(1+t)^{1/p}} \qquad \forall t \geq 0.$$

Proof Let us set

$$F(u) := \frac{1}{p+2} \int_{\mathbb{X}} |u(x)|^{p+2} \, d\mu(x).$$

We claim that

$$[\nabla F(u)](x) = |u(x)|^p u(x) \tag{4.5}$$

is the gradient of F in the sense of (2.3), and that all the assumptions of our abstract results (Theorem 2.2 and Theorem 2.3) are satisfied. All constants c_1, …, c_8 in the sequel depend only on $\mu(\mathbb{X})$, p, K_1, K_2, and on the coerciveness constant ν which appears in (2.13). Assumption (Hp1) is trivial, so that we can concentrate on the remaining ones.

Verification of (Hp2) Assumption (i), and the fact that $\mu(\mathbb{X}) < +\infty$, imply the following inclusions

$$D(A^{1/2}) \subseteq L^{2(p+1)}(\mathbb{X}, \mu) \subseteq L^{p+2}(\mathbb{X}, \mu) \subseteq L^2(\mathbb{X}, \mu). \tag{4.6}$$

Thus F is finite at least for every $u \in D(A^{1/2})$. Moreover, it is trivial that $F(0) = 0$ and $F(u) \geq 0$ for every $u \in D(A^{1/2})$.

Verification of (Hp3) Assumption (i) implies that $\nabla F(u)$, as defined by (4.5), is in H for every $u \in D(A^{1/2})$. Now we show that for every u and v in $D(A^{1/2})$ we have that

$$|F(u+v) - F(u) - \langle \nabla F(u), v \rangle|_H \leq c_1 \left(|u|^p_{D(A^{1/2})} + |v|^p_{D(A^{1/2})} \right) |v|_{D(A^{1/2})} \cdot |v|_H, \tag{4.7}$$

which clearly implies (2.3). To this end, we start from the inequality

$$\left| \frac{1}{p+2} \left(|a+b|^{p+2} - |a|^{p+2} \right) - |a|^p ab \right| \leq (p+1) \cdot 2^{p-1} \left(|a|^p + |b|^p \right) b^2 \qquad \forall (a,b) \in \mathbb{R}^2,$$

which follows from the second order Taylor's expansion of the function $|\sigma|^{p+2}$. Setting $a := u(x)$, $b := v(x)$, and integrating over $\mathbb{X}$, we obtain that

$$|F(u+v) - F(u) - \langle \nabla F(u), v\rangle|_H \leq c_2 \int_{\mathbb{X}} (|u(x)|^p + |v(x)|^p)\, |v(x)|^2\, dx. \tag{4.8}$$

From (4.3) we deduce

$$\left\|(|u|^p + |v|^p) \cdot v^2\right\|_{L^1(\mathbb{X},\mu)} \leq c_3 \left(|u|^p_{D(A^{1/2})} + |v|^p_{D(A^{1/2})}\right) \cdot |v|_{D(A^{1/2})} \cdot |v|_H.$$

Plugging this estimate into (4.8), we obtain (4.7).

Verification of (Hp4) We prove for every u and v in $D(A^{1/2})$ the inequality

$$|\nabla F(u) - \nabla F(v)|^2_H \leq c_4 \left(|u|^{2p}_{D(A^{1/2})} + |v|^{2p}_{D(A^{1/2})}\right) |u-v|^2_{D(A^{1/2})}, \tag{4.9}$$

which implies (2.4). To this end, we start from the inequality

$$\left||a|^p a - |b|^p b\right| \leq (p+1)\, (|a|^p + |b|^p)\, |a-b| \qquad \forall (a,b) \in \mathbb{R}^2,$$

which easily follows from the mean value theorem applied to the function $|\sigma|^p\sigma$. Setting $a := u(x)$, $b := v(x)$, and integrating over $\mathbb{X}$, we obtain that

$$\begin{aligned} |\nabla F(u) - \nabla F(v)|^2_H &= \int_{\mathbb{X}} \left||u(x)|^p u(x) - |v(x)|^p v(x)\right|^2 dx \\ &\leq c_5 \int_{\mathbb{X}} \left(|u(x)|^{2p} + |v(x)|^{2p}\right) |u(x) - v(x)|^2\, dx. \end{aligned} \tag{4.10}$$

From (4.4) we infer

$$\left\|(|u|^{2p} + |v|^{2p}) \cdot |u-v|^2\right\|_{L^1(\mathbb{X},\mu)} \leq c_6 \left(|u|^{2p}_{D(A^{1/2})} + |v|^{2p}_{D(A^{1/2})}\right) \cdot |u-v|^2_{D(A^{1/2})}.$$

Plugging this estimate into (4.10), we obtain (4.9).

Verification of (Hp5) It is trivially satisfied.

Verification of (Hp6) Exploiting (4.6) once again, we find

$$|u|^{p+2}_H = \|u\|^{p+2}_{L^2(\mathbb{X},\mu)} \leq c_7 \|u\|^{p+2}_{L^{p+2}(\mathbb{X},\mu)} = c_7(p+2)F(u)$$

for every $u \in D(A^{1/2})$, which proves (2.9).

Verification of assumption (2.14) From (4.6) we find

$$|\nabla F(u)|_H = |u|^{(p+1)}_{L^{2(p+1)}(\mathbb{X},\mu)} \leq c_8 |u|^{(p+1)}_{D(A^{1/2})}$$

for every $u \in D(A^{1/2})$, which proves (2.14) with $\alpha = p$ for any $\rho > 0$. $\square$

We are finally ready to apply our theory to hyperbolic partial differential equations. We concentrate on the model examples presented in the introduction. We recall that in the Dirichlet case even the existence of a single slow solution was an open problem. Also in the Neumann case, where existence of slow solutions was already known, the method of this paper gives the explicit conditions (3.17) for a solution to decay slowly, conditions which were not known before.

Theorem 4.2 (Neumann problem) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with the cone property. Let p be a positive exponent, with no further restriction if $n \in \{1, 2\}$, and $p \leq 2/(n-2)$ if $n \geq 3$.*

Let us consider the damped hyperbolic equation

$$u_{tt}(t,x) + u_t(t,x) - \Delta u(t,x) + |u(t,x)|^p u(t,x) = 0 \qquad \forall (t,x) \in [0,+\infty) \times \Omega, \quad (4.11)$$

with homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial n}(t,x) = 0 \qquad \forall (t,x) \in [0,+\infty) \times \partial\Omega, \quad (4.12)$$

and initial data

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \qquad \forall x \in \Omega. \quad (4.13)$$

Then we have the following conclusions.

(1) (Decay for all weak solutions) *For every $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$, problem (4.11) through (4.13) has a unique global weak solution*

$$u \in C^0\left([0,+\infty); H^1(\Omega)\right) \cap C^1\left([0,+\infty); L^2(\Omega)\right).$$

Moreover there exists a constant M_1 such that

$$\|u(t)\|_{L^2(\Omega)} \leq \frac{M_1}{(1+t)^{1/p}} \qquad \forall t \geq 0. \quad (4.14)$$

(2) (Existence of slow solutions) *There exist a nonempty open set $\mathcal{S} \subseteq H^1(\Omega) \times L^2(\Omega)$, and positive constants M_2 and M_3, with the following property. For every pair of initial conditions $(u_0, u_1) \in \mathcal{S}$, the unique global weak solution of problem (4.11) through (4.13) satisfies*

$$\frac{M_2}{(1+t)^{1/p}} \leq \|u(t)\|_{L^2(\Omega)} \leq \frac{M_3}{(1+t)^{1/p}} \qquad \forall t \geq 0. \quad (4.15)$$

Theorem 4.3 (Dirichlet problem) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with the cone property, and let λ_1 be the first eigenvalue of $-\Delta$ in Ω, with Dirichlet boundary conditions. Let p be a positive exponent, with no further restriction if $n \in \{1, 2\}$, and $p \leq 2/(n-2)$ if $n \geq 3$.*

Let us consider the damped hyperbolic equation

$$u_{tt}(t, x) + u_t(t, x) - \Delta u(t, x) - \lambda_1 u(t, x) + |u(t, x)|^p u(t, x) = 0 \tag{4.16}$$

in $[0, +\infty) \times \Omega$, with homogeneous Dirichlet boundary conditions

$$u(t, x) = 0 \qquad \forall (t, x) \in [0, +\infty) \times \partial\Omega, \tag{4.17}$$

and initial data (4.13).

Then we have the following conclusions.

(1) (Decay for all weak solutions) *For every $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, problem (4.16), (4.17), (4.13) has a unique global weak solution*

$$u \in C^0\left([0, +\infty); H_0^1(\Omega)\right) \cap C^1\left([0, +\infty); L^2(\Omega)\right)$$

satisfying (4.14).

(2) (Existence of slow solutions) *There exist a nonempty open set $\mathcal{S} \subseteq H_0^1(\Omega)) \times L^2(\Omega)$, and positive constants M_2 and M_3, with the following property. For every pair of initial conditions $(u_0, u_1) \in \mathcal{S}$, the unique global weak solution of problem (4.16), (4.17), (4.13) satisfies (4.15).*

Proof of Theorem 4.2 and Theorem 4.3 We plan to apply Theorem 4.1 with $\mathbb{X} = \Omega$, and μ equal to the Lebesgue measure on Ω. Concerning the operator A, we distinguish two cases.

- In the case of Theorem 4.2 the operator is $Au = -\Delta u$ with Neumann boundary conditions, so that $D(A) = H^2(\Omega)$, $D(A^{1/2}) = H^1(\Omega)$, and $\ker A \neq \{0\}$ because it consists of all (locally) constant functions.

- In the case of Theorem 4.3 the operator is $Au = -\Delta u - \lambda_1 u$ with Dirichlet boundary conditions, so that $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $D(A^{1/2}) = H_0^1(\Omega)$, and $\ker A \neq \{0\}$ because it consists of the first eigenspace of $-\Delta$.

In both cases, the norms $|u|_H$ and $|u|_{D(A^{1/2})}$ are equivalent to the norms $\|u\|_{L^2(\Omega)}$ and $\|u\|_{H^1(\Omega)}$, respectively, and the coerciveness assumption (2.13) is satisfied because Ω is bounded and eigenvalues are an increasing sequence.

Now we proceed to the verification of the assumptions of Theorem 4.1, which is the same in both cases. The cone property and the boundedness of Ω guarantee the usual Sobolev embeddings

$$H^1(\Omega) \subseteq L^q(\Omega) \begin{cases} \forall q < +\infty & \text{if } n \leq 2, \\ \forall q \leq 2^* = 2n/(n-2) & \text{if } n \geq 3. \end{cases} \tag{4.18}$$

All constants c_1, c_2, c_3 in the sequel depend only on p, and on the Sobolev constants.

Verification of (4.2) It follows from (4.18) with $q = 2(p+1)$ (note that our assumption on p is equivalent to $2(p+1) \leq 2^*$ if $n \geq 3$).

Verification of (4.3) Let u and v be in $D(A^{1/2})$. If $n \leq 2$, we apply Hölder's inequality with three terms and exponents 4, 4, 2, and we obtain

$$\big\| |u|^p \cdot v \cdot v \big\|_{L^1(\Omega)} \leq \|u\|_{L^{4p}(\Omega)}^p \cdot \|v\|_{L^4(\Omega)} \cdot \|v\|_{L^2(\Omega)}.$$

Thus from (4.18) with $q = 4p$ and $q = 4$ we conclude that

$$\big\| |u|^p \cdot v \cdot v \big\|_{L^1(\Omega)} \leq \|u\|_{H^1(\Omega)}^p \cdot \|v\|_{H^1(\Omega)} \cdot \|v\|_{L^2(\Omega)},$$

which is exactly (4.3). If $n \geq 3$, we apply Hölder's inequality with three terms and exponents n, 2^*, 2, and we obtain

$$\big\| |u|^p \cdot v \cdot v \big\|_{L^1(\Omega)} \leq \|u\|_{L^{np}(\Omega)}^p \cdot \|v\|_{L^{2^*}(\Omega)} \cdot \|v\|_{L^2(\Omega)}.$$

Thus from (4.18) with $q = np$ (note that $np \leq 2^*$) and $q = 2^*$ we conclude that

$$\big\| |u|^p \cdot v \cdot v \big\|_{L^1(\Omega)} \leq c_1 \|u\|_{H^1(\Omega)}^p \cdot \|v\|_{H^1(\Omega)} \cdot \|v\|_{L^2(\Omega)},$$

which proves (4.3) also in the case $n \geq 3$.

Verification of (4.4) Let u and v be in $D(A^{1/2})$. If $n \leq 2$, we apply Hölder's inequality with exponents 2 and 2, and then (4.18). We derive

$$\big\| |u|^{2p} \cdot v^2 \big\|_{L^1(\Omega)} \leq \|u\|_{L^{4p}(\Omega)}^{2p} \cdot \|v\|_{L^4(\Omega)}^2 \leq c_2 \|u\|_{H^1(\Omega)}^{2p} \cdot \|v\|_{H^1(\Omega)}^2,$$

which proves (4.4) in this case. If $n \geq 3$, we apply Hölder's inequality with exponents $n/2$ and $n/(n-2)$, and then (4.18). Since $np \leq 2^*$, we find

$$\big\| |u|^{2p} \cdot v^2 \big\|_{L^1(\Omega)} \leq \|u\|_{L^{np}(\Omega)}^{2p} \cdot \|v\|_{L^{2^*}(\Omega)}^2 \leq c_3 \|u\|_{H^1(\Omega)}^{2p} \cdot \|v\|_{H^1(\Omega)}^2,$$

which proves (4.4) also in the case $n \geq 3$. □

Remark 4.4 For the sake of simplicity and shortness, we limited ourselves to the model nonlinearity $g_p(\sigma) = |\sigma|^p \sigma$. On the other hand, all results can be easily extended, with standard adjustments (such as the restriction to L^∞-small initial data in low dimension), to equations with nonlinear terms which behave as $g_p(\sigma)$ just in a neighborhood of the origin.

4.2 Some nonlocal equations involving projection operators

The following result is suited to nonlocal partial differential equations where a power nonlinearity is applied to some integral of the unknown, and not to the unknown itself.

Theorem 4.5 (Semi-abstract result for nonlocal equations) *Let H be a Hilbert space, and let A be a linear operator on H with dense domain $D(A)$ satisfying assumptions (2.12) and (2.13) of Theorem 2.3, and such that*

$$\dim(\ker A) < +\infty. \tag{4.19}$$

Let M be a closed vector subspace of H such that

$$M^{\perp} \cap \ker A = \{0\}, \tag{4.20}$$

where $M^{\perp}$ denotes the space orthogonal to M. Let $P_M : H \to M$ denote the orthogonal projection. Let $p > 0$, and let us consider the second order equation

$$u''(t) + u'(t) + Au(t) + |P_M u(t)|^p P_M u(t) = 0. \tag{4.21}$$

Then we have the following conclusions.

(1) (Decay for all weak solutions) *For every $(u_0, u_1) \in D(A^{1/2}) \times H$, problem (4.21), (2.2) has a unique global weak solution with the regularity prescribed by (2.5). Moreover there exists a constant M_1 such that*

$$|u(t)| \leq \frac{M_1}{(1+t)^{1/p}} \qquad \forall t \geq 0.$$

(2) (Existence of slow solutions) *There exist a nonempty open set $\mathcal{S} \subseteq D(A^{1/2}) \times H$, and positive constants M_2 and M_3, with the following property. For every pair of initial conditions $(u_0, u_1) \in \mathcal{S}$, the unique global weak solution of problem (4.21)–(2.2) satisfies*

$$\frac{M_2}{(1+t)^{1/p}} \leq |u(t)| \leq \frac{M_3}{(1+t)^{1/p}} \qquad \forall t \geq 0.$$

Proof Let us set

$$F(u) := \frac{1}{p+2} |P_M u|^{p+2}.$$

We claim that

$$[\nabla F(u)](x) = |P_M u|^p P_M u$$

is the gradient of F in the sense of (2.3), and that all the assumptions of our abstract results (Theorem 2.2 and Theorem 2.3) are satisfied. Constants c_1, c_2, c_3 in the sequel depend only on the operator A, on the subspace M, on p, and on the coerciveness constant ν which appears in (2.13).

Assumptions (Hp1) and (Hp2) are trivial in this case.

Assumptions (Hp3) and (Hp4) require a completely standard verification, based on the simple fact that the real function $|\sigma|^p\sigma$ is of class C^1 when $p > 0$. We omit the details for the sake of shortness.

Assumption (Hp5) follows from the equality

$$\langle \nabla F(u), u\rangle = |P_M u|^p \langle P_M u, u\rangle = |P_M u|^p |P_M u|^2 \qquad \forall u \in H.$$

We have now to verify (Hp6). This requires three steps. Let $P : H \to \ker A$ denote the orthogonal projection on $\ker A$. The first step is just observing that assumption (2.13) is equivalent to

$$|A^{1/2}u|^2 \geq \nu|u - Pu|^2 \qquad \forall u \in D(A^{1/2}). \tag{4.22}$$

The second step consists in proving that there exists $c_1 > 0$ such that

$$|v|^2 \leq c_1|P_M v|^2 \qquad \forall v \in \ker A. \tag{4.23}$$

To this end we set

$$c_2 := \min\left\{|P_M v|^2 : v \in \ker A,\ |v| = 1\right\},$$

and we observe that the minimum exists because of assumption (4.19), and it is positive because of assumption (4.20). This is enough to prove that (4.23) holds true with $c_1 = c_2^{-1}$.

Applying (4.23) with $v := Pu$, we obtain

$$\begin{aligned}
|Pu|^2 &\leq c_1|P_M(Pu)|^2 \\
&= c_1|P_M u - P_M(u - Pu)|^2 \\
&\leq 2c_1|P_M u|^2 + 2c_1|P_M(u - Pu)|^2 \\
&\leq 2c_1|P_M u|^2 + 2c_1|u - Pu|^2.
\end{aligned}$$

If $u \in D(A^{1/2})$, we can now apply (4.22) and conclude that

$$\begin{aligned}
|u|^2 = |Pu|^2 + |u - Pu|^2 &\leq 2c_1|P_M u|^2 + (2c_1 + 1)|u - Pu|^2 \\
&\leq 2c_1|P_M u|^2 + \frac{2c_1 + 1}{\nu}|A^{1/2}u|^2,
\end{aligned}$$

hence

$$|u|^{p+2} \leq c_3\left(|P_M u|^{p+2} + |A^{1/2}u|^{p+2}\right) \leq c_3\left(p + 2 + |A^{1/2}u|^p\right)(F(u) + |A^{1/2}u|^2),$$

which proves (Hp6).

Finally, (2.14) is obviously satisfied with $R = 1$, independently of ρ and α. $\Box$

We conclude with two examples of application of Theorem 4.5. In a certain sense they represents the two extremes, namely the case where $M = H$, hence as large as possible, and the case where M is one-dimensional. We omit the simple proofs.

Theorem 4.6 *Let $\Omega \subseteq \mathbb{R}^n$ and p be as in Theorem 4.2. Let us consider the integro-differential damped hyperbolic equation*

$$u_{tt}(t,x) + u_t(t,x) - \Delta u(t,x) + \left(\int_\Omega u^2(t,x)\,dx\right)^{p/2} u(t,x) = 0,$$

in $[0,+\infty) \times \Omega$, with Neumann boundary conditions (4.12), and initial data (4.13).

Then we have the same conclusions as those of Theorem 4.2.

Theorem 4.7 *Let $\Omega \subseteq \mathbb{R}^n$ and p be as in Theorem 4.2. Let $\{\Omega_i\}_{i \in I}$ be the set of all connected components of Ω, and let $\varphi \in H^1(\Omega)$ be a function such that*

$$\int_{\Omega_i} \varphi(x)\,dx \neq 0 \qquad \forall i \in I. \tag{4.24}$$

Let $g_p : \mathbb{R} \to \mathbb{R}$ be defined by $g_p(\sigma) := |\sigma|^p \sigma$ for every $\sigma \in \mathbb{R}$, and let us consider the integro-differential damped hyperbolic equation

$$u_{tt}(t,x) + u_t(t,x) - \Delta u(t,x) + g_p\left(\int_\Omega u(t,x)\varphi(x)\,dx\right)\varphi(x) = 0,$$

in $[0,+\infty) \times \Omega$, with Neumann boundary conditions (4.12), and initial data (4.13).

Then we have the same conclusions as those of Theorem 4.2.

One can state similar results also for the Dirichlet problem, namely by replacing the nonlinear term in Theorem 4.3 with the nonlinear terms appearing in Theorem 4.6 or Theorem 4.7. The only difference is that in the Dirichlet case the non-orthogonality condition (4.24) becomes

$$\int_\Omega \varphi(x)e(x)\,dx \neq 0$$

for every nonzero function $e(x)$ in the first eigenspace of the Dirichlet Laplacian.

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